International Journal of Mathematics and Computation

ISSN 0974-5718 (Print); ISSN 0974-570X (Online)

2011, Volume 10, Number M11 (Special Volume)

Open Access Subscription or Fee Access

First Special Volume on The 6th International Conference: Dynamical Systems and Applications-2010, Antalya–Turkey, July 10-14, 2010

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Impulsive Cohen-Grossberg Neural Networks with S-Type Distributed Delays and Reaction-Diffusion Terms

Haydar Akça, Valéry Covachev, Zlatinka Covacheva

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Keywords

Cohen-Grossberg neural networks, S-type delays, impulses, reaction-diffusion.

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Abstract

An impulsive Cohen-Grossberg neural network with time-varying and S-type distributed delays and reaction-diffusion terms is considered. Under suitable conditions in terms of $M-$matrices it is proved that the system has a unique equilibrium point which is globally exponentially stable.
AMS 2000 Subject Classification: 34A37, 35B35, 35K57, 92B20.
Key Words: Cohen-Grossberg neural networks, S-type delays, impulses, reaction-diffusion.

1 Introduction

Since Cohen-Grossberg neural networks were proposed by Cohen and Grossberg [4] in 1983, extensive work has been done on this subject due to their extensive applications in classification of patterns, associative memories, image processing, quadratic optimization, and other areas. In implementation of neural networks, however, time delays inevitably occur due to the finite switching speed of neurons and amplifiers.

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely dicrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems [7].

It is well known that diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields [14], so the activations must be considered to vary in space as well as in time. The papers [12, 13] are devoted to the exponential stability of impulsive Cohen-Grossberg neural networks with, respectively, time-varying and distributed delays and reaction-diffusion terms.

In the present paper we consider an impulsive Cohen-Grossberg neural network with both time-varying and S-type distributed delays [2, 8, 11, 17] and reaction-diffusion terms as in [16] which are of a form more general than in [12, 13], and zero Neumann boundary conditions. Under suitable conditions in terms of $M-$matrices it is proved that the system obtained has a unique equilibrium point which is globally exponentially stable.

2 Model description and preliminaries

We consider the following system of impulsive Cohen-Grossberg neural networks with time-varying and distributed delays and reaction-diffusion terms, and zero Neumann boundary conditions:

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} &= \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(D_i(t,x)\frac{\partial u_i(t,x)}{\partial x_k}\right)+\alpha_i(u_i(t,x))\Bigg[-\beta_i(u_i(t,x))\\ &+ \sum_{j=1}^{m}a_{ij}f_j(u_j(t,x))+\sum_{j=1}^{m}b_{ij}g_j(u_j(t-\tau_{ij}(t),x)) \qquad (1)\\ &+ \sum_{j=1}^{m}c_{ij}\int_{-\infty}^{0}h_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)+J_i\Bigg],\quad t>0,\quad t\neq t_k,\end{aligned}$$

$$\Delta u_i(t_k,x)=-B_{ik}u_i(t_k,x)+\int_{t_{k-1}-t_k}^{0}u_i(t_k+\theta)\,d\zeta_k(\theta)+\gamma_{ik},\quad k\in\mathbb{N},$$

$$\left.\frac{\partial u_i}{\partial\nu}\right|_{\partial\Omega}=0,\qquad u_i(s,x)=\phi_i(s,x),\quad s\le 0,\quad x\in\Omega,\quad i=\overline{1,m},$$

where $m\geq 2$ is the number of neurons in the network; Ω is a bounded compact set with smooth boundary $\partial\Omega$ and $\operatorname{mes}\Omega>0$; $\partial/\partial\nu$ is the outward normal derivative; $D_i(t,x)>0$ are smooth functions corresponding to the transmission diffusion operator along the $i-$th neuron; $\alpha_i(u_i)$ represent amplification functions; $\beta_i(u_i)$ are appropriately behaving functions which support the stabilizing feedback term $-\alpha_i(u_i)\beta_i(u_i)$ of the $i-$th neuron; a_{ij}, b_{ij}, c_{ij} denote the connection weights (or strengths) of the synaptic connections between the $j-$th neuron and the $i-$th neuron; $f_j(u_j)$, $g_j(u_j)$, $h_j(u_j)$ denote the activation functions of the $j-$th neuron; J_i denotes external input to the $i-$th neuron; $\tau_{ij}(t)$ $(i,j=\overline{1,m})$ correspond to the transmission delays; the past effect of the $j-$th neuron on the $i-$th neuron is given by the Lebesgue-Stieltjes integral $\int_{-\infty}^{0} h_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)$; $\Delta u_i(t_k,x) = u_i(t_k+0,x) - u_i(t_k-0,x)$ denote impulsive state displacements at fixed moments (instants) of time t_k, $k\in\mathbb{N}$, involving Lebesgue-Stieltjes integrals. Here it is assumed that $u_i(t_k-0,x)$ and $u_i(t_k+0,x)$ denote respectively the left-hand and right-hand limit at t_k, and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0=t_0<t_1<t_2<\cdots<t_k\to+\infty$ as $k\to+\infty$.

As usual in the theory of impulsive differential equations (and unlike [12, 13]), at the points of discontinuity t_k of the solution $t\mapsto u_i(t,x)$ we assume that $u_i(t_k,x)\equiv u_i(t_k-0,x)$. It is clear that, in general, the derivatives $\frac{\partial u}{\partial t}(t_k,x)$ do not exist. On the other hand, according to the first equality of (1), there do exist the limits $\frac{\partial u}{\partial t}(t_k\mp 0,x)$. According to the above convention, we assume $\frac{\partial u}{\partial t}(t_k,x)\equiv\frac{\partial u}{\partial t}(t_k-0,x)$.

Throughout the paper we assume that:

A1 The amplification functions $\alpha_i:\mathbb{R}\to(0,+\infty)$ are continuous and bounded in the sense that $0<\underline{\alpha}_i\le\alpha_i(u)\le\overline{\alpha}_i$ for $u\in\mathbb{R}$, $i=\overline{1,m}$.

A2 The stabilizing functions $\beta_i:\mathbb{R}\to\mathbb{R}$ are continuous and monotone increasing, namely, $0<\underline{\beta}_i\le\dfrac{\beta_i(u)-\beta_i(v)}{u-v}$ for $u,v\in\mathbb{R}$, $u\ne v$, $i=\overline{1,m}$.

A3 For the activation functions $f_i(u),g_i(u),h_i(u)$ there exist positive constants F_i,G_i,H_i such that $F_i=\sup\limits_{u\ne v}\left|\dfrac{f_i(u)-f_i(v)}{u-v}\right|$, $G_i=\sup\limits_{u\ne v}\left|\dfrac{g_i(u)-g_i(v)}{u-v}\right|$, $H_i=\sup\limits_{u\ne v}\left|\dfrac{h_i(u)-h_i(v)}{u-v}\right|$ for all $u,v\in\mathbb{R}$, $u\ne v$, $i=\overline{1,m}$.

A4 $\tau_{ij}(t)$ satisfy $0\le\tau_{ij}(t)\le\tau_{ij}$, $0\le\dot{\tau}_{ij}(t)\le\delta<1$ $(i,j=\overline{1,m})$.

A5 $\eta_{ij}(\theta)$ $(i,j=\overline{1,m})$, $\zeta_k(\theta)$ $(k\in\mathbb{N})$ are nondecreasing bounded variation functions on $(-\infty,0]$ and $[t_{k-1}-t_k,0]$, respectively, and $\int_{-\infty}^{0}e^{-\lambda\theta}\,d\eta_{ij}(\theta)=k_{ij}(\lambda)$ are continuous functions on $[0,\lambda_0]$ for some $\lambda_0>0$ and $k_{ij}(0)=1$ (without loss of generality), $\int_{t_{k-1}-t_k}^{0}d\zeta_k(\theta)=\omega_k$.

The components of an equilibrium point $u^*=(u_1^*,\dots,u_m^*)$ of system (1) are governed by the algebraic system

$$-\beta_i(u_i^*)+\sum_{j=1}^{m}\left(a_{ij}f_j(u_j^*)+b_{ij}g_j(u_j^*)+c_{ij}h_j(u_j^*)\right)+J_i=0,\qquad i=\overline{1,m}.\quad(2)$$

and satisfy the linear equations

$$(-B_{ik}+\omega_k)u_i^*+\gamma_{ik}=0, \qquad i=\overline{1,m}, \quad k\in\mathbb{N}. \tag{3}$$

We assume that

A6 The linear equations (3) are satisfied for any solution u^* of system (2).

Denote

$$\|u_i(t,\cdot)\| = \left(\int_\Omega \big(u_i(t,x)\big)^2\,dx\right)^{1/2}.$$

Definition 1 An equilibrium point $u^*=(u_1^*,\dots,u_m^*)^T$ of system (1) is said to be *globally exponentially stable* (with Lyapunov exponent λ) if there exist constants $\lambda>0$ and $M\geq 1$ such that for any solution $u(t,x)=(u_1(t,x),\dots,u_m(t,x))^T$ of system (1) we have

$$\sum_{i=1}^m \|u_i(t,\cdot)-u_i^*\| \leq M\sup_{s\leq 0}\sum_{i=1}^m \|\phi_i(s,\cdot)-u^*\|e^{-\lambda t} \quad \text{for all } \ t\geq 0.$$

Definition 2 [3] A real matrix $A=(a_{ij})_{m\times m}$ is said to be an $M-$matrix if $a_{ij}\leq 0$ for $i,j=\overline{1,m},\ i\neq j$ and all successive principle minors of A are positive.

Lemma 1 [3] *Let $A=(a_{ij})_{m\times m}$ be a real matrix with non-positive off-diagonal elements. Then A is an $M-$matrix if and only if one of the following conditions holds:*

(1) *There exists a vector $\xi=(\xi_1,\xi_2,\dots,\xi_m)^T$ with $\xi_i>0$ such that every component of ξ^TA is positive — that is, $\sum\limits_{i=1}^m \xi_i a_{ij}>0$, $j=\overline{1,m}$.*

(2) *There exists a vector $\xi=(\xi_1,\xi_2,\dots,\xi_m)^T$ with $\xi_i>0$ such that every component of $A\xi$ is positive — that is, $\sum\limits_{j=1}^m a_{ij}\xi_j>0$, $i=\overline{1,m}$.*

For more details about $M-$matrices the reader is referred to [5, 10].

Further on we will need the following lemma.

Lemma 2 [6] *A locally invertible C^0 map $\Phi:\ \mathbb{R}^m\to\mathbb{R}^m$ is a homeomorphism of $\mathbb{R}^m$ onto itself if and only if it is proper.*

In fact, this assertion is due to Hadamard [9]. A mapping is proper if the preimage of every compact is compact. In the finite-dimensional case it suffices to show that $\|\Phi(x)\|\to\infty$ as $\|x\|\to\infty$.

Now let us introduce the following matrices:

$$\begin{array}{lll}\underline{\alpha}=\mathrm{diag}(\underline{\alpha}_1,\dots,\underline{\alpha}_m), & \overline{\alpha}=\mathrm{diag}(\overline{\alpha}_1,\dots,\overline{\alpha}_m), & \underline{\beta}=\mathrm{diag}(\underline{\beta}_1,\dots,\underline{\beta}_m),\\ F=\mathrm{diag}(F_1,\dots,F_m), & G=\mathrm{diag}(G_1,\dots,G_m), & H=\mathrm{diag}(H_1,\dots,H_m),\\ |A|=(|a_{ij}|)_{m\times m}, & |B|=(|b_{ij}|)_{m\times m}, & |C|=(|c_{ij}|)_{m\times m}.\end{array}$$

Lemma 3 *Let assumptions* **A2**, **A3**, **A5**, **A6** *hold and suppose* $\underline{\beta} - \big(|A|F + |B|G + |C|H\big)$ *is an* $M-$*matrix. Then system* (1) *has a unique equilibrium point.*

Proof. Let us define a mapping $\Phi:\ \mathbb{R}^m \to \mathbb{R}^m$ by $\Phi(u) = (\Phi_1(u), \Phi_2(u), \ldots, \Phi_m(u))^T$ for $u \in \mathbb{R}^m$, where

$$\Phi_i(u) = -\beta_i(u_i) + \sum_{j=1}^{m}\big[a_{ij}f_j(u_j) + b_{ij}g_j(u_j) + c_{ij}h_j(u_j)\big] + J_i, \quad i = \overline{1,m}.$$

The space $\mathbb{R}^m$ is endowed with the norm $\|u\| = \sum\limits_{i=1}^{m}|u_i|$. Under the assumptions **A2**, **A3**, $\Phi(u) \in C^0$. It is known that if $\Phi(u) \in C^0$ is a homeomorphism of $\mathbb{R}^m$, then there is a unique point $u^* = (u_1^*, u_2^*, \ldots, u_m^*)^T \in \mathbb{R}^m$ such that $\Phi(u^*) = 0$, that is, $\Phi_i(u^*) = 0$, $i = \overline{1,m}$. The last equalities are, in fact, (2), so by **A5** $u^* = (u_1^*, u_2^*, \ldots, u_m^*)^T$ is the equilibrium point we are looking for.

To demonstrate the one-to-one property of $\Phi(u)$, we take arbitrary vectors $u, v \in \mathbb{R}^m$ and assume that $\Phi(u) = \Phi(v)$. From

$$\begin{aligned}\beta_i(u_i) - \beta_i(v_i) &= \sum_{j=1}^{m}\big[a_{ij}\left(f_j(u_j) - f_j(v_j)\right)\\ + \quad b_{ij}\left(g_j(u_j) - g_j(v_j)\right) &+ c_{ij}\left(h_j(u_j) - h_j(v_j)\right)\big], \quad i = \overline{1,m},\end{aligned}$$

under the assumptions **A2**, **A3** one obtains

$$\underline{\beta}_i|u_i - v_i| \le \sum_{j=1}^{m}\big[|a_{ij}|F_j + |b_{ij}|G_j + |c_{ij}|H_j\big]|u_j - v_j|, \qquad i = \overline{1,m}.$$

We note that the last inequalities can be written in a matrix form as

$$\big[\underline{\beta} - \big(|A|F + |B|G + |C|H\big)\big]|u - v| \le 0,$$

which implies $u - v$=0.

Next we show that $\|\Phi(u)\| \to \infty$ as $\|u\| \to \infty$. It suffices to show that $\|\tilde{\Phi}(u)\| \to \infty$, where $\tilde{\Phi}(u) = \Phi(u) - \Phi(0)$. We have $\tilde{\Phi}(u) = (\tilde{\Phi}_1(u), \tilde{\Phi}_2(u), \ldots, \tilde{\Phi}_m(u))^T$, where

$$\begin{aligned}&\tilde{\Phi}_i(u) = -(\beta_i(u_i) - \beta_i(0))\\ + \quad &\sum_{j=1}^{m}\big[a_{ij}(f_j(u_j) - f_j(0)) + b_{ij}(g_j(u_j) - g_j(0)) + c_{ij}(h_j(u_j) - h_j(0))\big]\end{aligned}$$

for $i = \overline{1,m}$. These equalities imply

$$\left|\tilde{\Phi}_i(x)\right| \ge \underline{\beta}_i|u_i| - \sum_{j=1}^{m}\big[|a_{ij}|F_j + |b_{ij}|G_j + |c_{ij}|H_j\big]|u_j|. \tag{4}$$

From the assumptions of Lemma 3 there exists a vector $\xi > 0$ such that

$$\underline{\beta}_j \xi_j - \sum_{i=1}^{m} \left[|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \right] \xi_i > 0, \qquad j = \overline{1,m}.$$

Let us denote

$$\mu = \min_{j=\overline{1,m}} \left\{ \underline{\beta}_j - \sum_{i=1}^{m} \left[|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \right] \frac{\xi_i}{\xi_j} \right\}.$$

Then from (4)

$$\begin{aligned}
\sum_{i=1}^{m} \xi_i \left| \tilde{\Phi}_i(u) \right| \quad & \geq \quad \sum_{i=1}^{m} \xi_i \left\{ \underline{\beta}_i |u_i| - \sum_{j=1}^{m} \left[|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \right] |u_j| \right\} \\
& = \quad \sum_{j=1}^{m} \xi_j \left\{ \underline{\beta}_j - \sum_{i=1}^{m} \left[|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \right] \frac{\xi_i}{\xi_j} \right\} |u_j| \\
& \geq \quad \mu \sum_{j=1}^{m} \xi_j |u_j|.
\end{aligned}$$

From the last inequality we deduce

$$\max_{i=\overline{1,m}} \xi_i \sum_{i=1}^{m} \left| \tilde{\Phi}_i(u) \right| \geq \sum_{i=1}^{m} \xi_i \left| \tilde{\Phi}_i(u) \right| \geq \mu \sum_{j=1}^{m} \xi_j |u_j| \geq \mu \min_{j=\overline{1,m}} \xi_j \sum_{j=1}^{m} |u_j|,$$

that is,

$$\|\tilde{\Phi}(u)\| = \sum_{i=1}^{m} \left| \tilde{\Phi}_i(u) \right| \geq \mu \frac{\min\limits_{i=\overline{1,m}} \xi_i}{\max\limits_{i=\overline{1,m}} \xi_i} \sum_{i=1}^{m} |u_i| = \mu \frac{\min\limits_{i=\overline{1,m}} \xi_i}{\max\limits_{i=\overline{1,m}} \xi_i} \|u\|.$$

The last inequality shows that $\|\tilde{\Phi}(u)\| \to \infty$ as $\|u\| \to \infty$.

According to Lemma 2, $\Phi(u) \in C^0$ is a homeomorphism of $\mathbb{R}^m$. $\square$

3 Main results

Theorem 1 *Let system* (1) *satisfy assumptions* **A1**–**A6**. *If there exists a vector* $\xi = (\xi_1, \ldots, \xi_m)^T > 0$ *and a number* $\lambda > 0$ *such that*

$$\sum_{i=1}^{m} \left\{ (\lambda - \underline{\alpha}_i \underline{\beta}_i) \delta_{ij} + \overline{\alpha}_i \left[|a_{ij}| F_j + |b_{ij}| G_j \frac{e^{\lambda \tau_{ij}}}{1-\delta} + |c_{ij}| H_j k_{ij}(\lambda) \right] \right\} \xi_i < 0 \quad (5)$$

for $j = \overline{1,m}$, *where* $\delta_{ii} = 1$, $\delta_{ij} = 0$ *for* $j \neq i$, *then system* (1) *has a unique equilibrium point* $u^* = (u_1^*, \ldots, u_m^*)^T$ *and there exists a constant* $M \geq 1$ *such* *that for any solution $u(t,x) = (u_1(t,x),\dots,u_m(t,x))^T$ of system* (1) *we have*

$$\begin{aligned}\sum_{i=1}^{m}\|u_i(t,\cdot)-u_i^*\| &\le Me^{-\lambda t}\prod_{k=1}^{i(0,t)}\left(\max_{i=\overline{1,m}}|1-B_{ik}|+\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\,d\zeta_k(\theta)\right)\\ &\times \sum_{i=1}^{m}\sup_{s\le 0}\|u_i(s,\cdot)-u_i^*\|, \qquad t\ge 0, \qquad (6)\end{aligned}$$

where $i(0,t) = \max\{k\in\{0\}\cup\mathbb{N}:\ t_k<t\}$ is the number of instants of impulse effect t_k in the interval $(0,t)$.

Proof. First let us note that condition (5) holds if and only if $\mathcal{A}(\delta) = \underline{\alpha}\underline{\beta} - \overline{\alpha}\big(|A|F + \frac{1}{1-\delta}|B|G + |C|H\big)$ is an $M-$matrix. In fact, if $\mathcal{A}(\delta)$ is an $M-$matrix, from Lemma 1 there exists a vector $\xi > 0$ such that $\xi^T\big[-\underline{\alpha}\underline{\beta} + \overline{\alpha}(|A|F + \frac{1}{1-\delta}|B|G + |C|H)\big] < 0$. By continuity, there exists $\lambda > 0$ such that (5) holds. Conversely, if (5) holds for some $\lambda^* > 0$, then it still holds for all $\lambda \in [0,\lambda^*]$. For $\lambda = 0$, from Lemma 1 we deduce that $\mathcal{A}(\delta)$ is an $M-$matrix.

If $\mathcal{A}(\delta)$ is an $M-$matrix, by virtue of $0<\delta<1$, $\mathcal{A}(0) = \underline{\alpha}\underline{\beta} - \overline{\alpha}(|A|F + |B|G + |C|H)$ is also an $M-$matrix. The inequality $\overline{\alpha}^{-1}\underline{\alpha} \le E$ (E is the identity matrix) implies that $\underline{\beta} - \big(|A|F + |B|G + |C|H\big)$ is an $M-$matrix. Thus condition (5), by virtue of Lemma 3, ensures the existence of a unique equilibrium point $u^* = (u_1^*,\dots,u_m^*)^T$ for system (1). For any other solution $u(t,x) = (u_1(t,x),\dots,u_m(t,x))^T$ of system (1) denote

$$w_i(t,x) = u_i(t,x) - u_i^*, \qquad i=\overline{1,m}, \quad x\in\Omega.$$

Thus system (1) is transformed into

$$\begin{aligned}\frac{\partial w_i(t,x)}{\partial t} &= \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(D_i(t,x)\frac{\partial w_i(t,x)}{\partial x_k}\right) + \tilde{\alpha}_i(w_i(t,x))\Bigg[-\tilde{\beta}_i(w_i(t,x))\\ &+ \sum_{j=1}^{m} a_{ij}\tilde{f}_j(w_j(t,x)) + \sum_{j=1}^{m} b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),x)) \qquad (7)\\ &+ \sum_{j=1}^{m} c_{ij}\int_{-\infty}^{0}\tilde{h}_j(w_j(t+\theta,x))\,d\eta_{ij}(\theta)\Bigg], \quad t>0, \quad t\ne t_k,\end{aligned}$$

$$\Delta w_i(t_k,x) = -B_{ik}w_i(t_k,x) + \int_{t_{k-1}-t_k}^{0} w_i(t_k+\theta)\,d\zeta_k(\theta), \quad k\in\mathbb{N},$$

$$\left.\frac{\partial w_i}{\partial \nu}\right|_{\partial\Omega} = 0, \qquad w_i(s,x) = \phi_i(s,x) - u_i^*, \quad s\le 0, \quad x\in\Omega, \quad i=\overline{1,m},$$

where $\tilde{\alpha}_i(w_i) = \alpha_i(w_i + u_i^*)$,

$$\tilde{\beta}_i(w_i) = \beta_i(w_i+u_i^*) - \beta_i(u_i^*), \qquad \tilde{f}_j(w_j) = f_j(w_j+u_j^*) - f_j(u_j^*),$$
$$\tilde{g}_j(w_j) = g_j(w_j+u_j^*) - g_j(u_j^*), \qquad \tilde{h}_j(w_j) = h_j(w_j+u_j^*) - h_j(u_j^*).$$

We multiply the $i-$th differential equation in (7) by $w_i(t,x)$ and integrate over the domain Ω:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big(w_i(t,x)\big)^2\,dx = \int_\Omega \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(D_i(t,x)\frac{\partial w_i(t,x)}{\partial x_k}\right) w_i(t,x)\,dx$$

$$\begin{aligned}
- & \int_\Omega \tilde{\alpha}_i(w_i(t,x))\tilde{\beta}_i(w_i(t,x))w_i(t,x)\,dx \\
+ & \int_\Omega \tilde{\alpha}_i(w_i(t,x))w_i(t,x)\sum_{j=1}^{m} a_{ij}\tilde{f}_j(w_j(t,x))\,dx \\
+ & \int_\Omega \tilde{\alpha}_i(w_i(t,x))w_i(t,x)\sum_{j=1}^{m} b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),x))\,dx \\
+ & \int_\Omega \tilde{\alpha}_i(w_i(t,x))w_i(t,x)\sum_{j=1}^{m} c_{ij}\int_{-\infty}^{0}\tilde{h}_j(w_j(t+\theta,x))\,d\eta_{ij}(\theta)\,dx.
\end{aligned}$$

By the zero Neumann boundary conditions we have

$$\int_\Omega \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(D_i(t,x)\frac{\partial w_i(t,x)}{\partial x_k}\right) w_i(t,x)\,dx = -\sum_{k=1}^{n}\int_\Omega D_i(t,x)\left(\frac{\partial w_i(t,x)}{\partial x_k}\right)^2 \le 0.$$

Next we have

$$\int_\Omega \tilde{\alpha}_i(w_i(t,x))\tilde{\beta}_i(w_i(t,x))w_i(t,x)\,dx \ge \underline{\alpha}_i\underline{\beta}_i\int_\Omega \big(w_i(t,x)\big)^2\,dx = \underline{\alpha}_i\underline{\beta_i}\|w_i(t,\cdot)\|^2;$$

$$\begin{aligned}
& \int_\Omega \tilde{\alpha}_i(w_i(t,x))w_i(t,x)\sum_{j=1}^{m} a_{ij}\tilde{f}_j(w_j(t,x))\,dx \\
& \le \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|\int_\Omega |w_i(t,x)|\,F_j\,|w_j(t,x)|\,dx \\
\le \quad & \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\left(\int_\Omega \big(w_i(t,x)\big)^2\,dx\right)^{1/2}\times\left(\int_\Omega \big(w_j(t,x)\big)^2\,dx\right)^{1/2} \\
& = \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\|w_i(t,\cdot)\|\;\|w_j(t,\cdot)\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_\Omega \tilde{\alpha}_i(w_i(t,x))w_i(t,x)\sum_{j=1}^{m} b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),x))\,dx \\
& \le \overline{\alpha}_i\sum_{j=1}^{m}|b_{ij}|G_j\|w_i(t,\cdot)\|\;\|w_j(t-\tau_{ij}(t),\cdot)\|
\end{aligned}$$

and

$$\int_\Omega \tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^m c_{ij}\int_{-\infty}^0 \tilde{h}_j(w_j(t+\theta,x))\,d\eta_{ij}(\theta)\,dx$$

$$\leq \overline{\alpha}_i\sum_{j=1}^m |c_{ij}|H_j\|w_i(t,\cdot)\|\int_{-\infty}^0 \|w_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta).$$

Combining the above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w_i(t,\cdot)\|^2 \leq -\underline{\alpha}_i\underline{\beta}_i\|w_i(t,\cdot)\|^2 + \overline{\alpha}_i\sum_{j=1}^m\Big\{|a_{ij}|F_j\|w_i(t,\cdot)\|\;\|w_j(t,\cdot)\|$$

$$+\,|b_{ij}|G_j\|w_i(t,\cdot)\|\;\|w_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j\|w_i(t,\cdot)\|\int_{-\infty}^0\|w_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta)\Big\}$$

or

$$D^+\|w_i(t,\cdot)\| \leq -\underline{\alpha}_i\underline{\beta}_i\|w_i(t,\cdot)\| + \overline{\alpha}_i\sum_{j=1}^m\Big\{|a_{ij}|F_j\|w_j(t,\cdot)\| \tag{8}$$

$$+\quad |b_{ij}|G_j\|w_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j\int_{-\infty}^0\|w_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta)\Big\},$$

where D^+ denotes the upper right Dini derivative.

If we introduce the notation $y_i(t) = e^{\lambda t}\|w_i(t,\cdot)\|$, then from (8) we find

$$D^+y_i(t) \leq (\lambda - \underline{\alpha}_i\underline{\beta}_i)y_i(t) + \overline{\alpha}_i\sum_{j=1}^m\Big\{|a_{ij}|F_jy_j(t) \tag{9}$$

$$+\quad |b_{ij}|G_jy_j(t-\tau_{ij}(t))e^{\lambda\tau_{ij}(t)} + |c_{ij}|H_j\int_{-\infty}^0 e^{-\lambda\theta}y_j(t+\theta)\,d\eta_{ij}(\theta)\Big\}.$$

We consider a Lyapunov functional

$$\begin{aligned} V(t) \quad &= \quad \sum_{i=1}^m\Big\{y_i(t) + \overline{\alpha}_i\sum_{j=1}^m|b_{ij}|G_j\frac{e^{\lambda\tau_{ij}}}{1-\delta}\int_{t-\tau_{ij}(t)}^t y_j(s)\,ds \\ &+ \quad \overline{\alpha}_i\sum_{j=1}^m|c_{ij}|H_j\int_{-\infty}^0 e^{-\lambda\theta}\left(\int_{t+\theta}^t y_j(s)\,ds\right)\,d\eta_{ij}(\theta)\Big\}\xi_i,\end{aligned}$$

where λ and ξ_i, $i=\overline{1,m}$ are as in (5). We may suppose that $\lambda\in(0,\lambda_0)$.

We note that $V(t)>0$ for $t\geq 0$ and

$$V(0) = \sum_{i=1}^m\Big\{y_i(0) + \overline{\alpha}_i\sum_{j=1}^m|b_{ij}|G_j\frac{e^{\lambda\tau_{ij}}}{1-\delta}\int_{-\tau_{ij}(t)}^0 y_j(s)\,ds$$

$$
\begin{aligned}
&+ \quad \overline{\alpha}_i \sum_{j=1}^{m} |c_{ij}| H_j \int_{-\infty}^{0} e^{-\lambda\theta} \left(\int_{\theta}^{0} y_j(s)\, ds \right) \, d\eta_{ij}(\theta) \Bigg\} \xi_i \\
&= \quad \sum_{i=1}^{m} \Bigg\{ y_i(0)\, \xi_i + G_i \sum_{j=1}^{m} |b_{ji}| \frac{\overline{\alpha}_j e^{\lambda \tau_{ji}}}{1-\delta} \int_{-\tau_{ji}(t)}^{0} y_i(s)\, ds\, \xi_j \\
&+ \quad H_i \sum_{j=1}^{m} |c_{ji}| \overline{\alpha}_j \int_{-\infty}^{0} e^{-\lambda\theta} \left(\int_{\theta}^{0} y_i(s)\, ds \right) \, d\eta_{ji}(\theta)\, \xi_j \Bigg\} \\
&\leq \quad \sum_{i=1}^{m} \Bigg\{ \xi_i + G_i \sum_{j=1}^{m} |b_{ji}| \frac{\overline{\alpha}_j e^{\lambda \tau_{ji}} \tau_{ji}}{1-\delta} \xi_j \\
&+ \quad H_i \sum_{j=1}^{m} |c_{ji}| \overline{\alpha}_j \int_{-\infty}^{0} e^{-\lambda\theta} (-\theta)\, d\eta_{ji}(\theta)\, \xi_j \Bigg\} \sup_{s \leq 0} y_(s),
\end{aligned}
$$

thus

$$
V(0) \leq M \sum_{i=1}^{m} \sup_{s \leq 0} y_i(s) \tag{10}
$$

with

$$
\begin{aligned}
M \quad &= \quad \max_{i=\overline{1,m}} \Bigg\{ \xi_i + G_i \sum_{j=1}^{m} |b_{ji}| \frac{\overline{\alpha}_j e^{\lambda \tau_{ji}} \tau_{ji}}{1-\delta} \xi_j \\
&+ \quad H_i \sum_{j=1}^{m} |c_{ji}| \overline{\alpha}_j \int_{-\infty}^{0} e^{-\lambda\theta} (-\theta)\, d\eta_{ji}(\theta)\, \xi_j \Bigg\}.
\end{aligned}
$$

The above integral is convergent because of $\lambda < \lambda_0$.

Calculating the rate of change of V(t) along the solutions of (7), by virtue of (9), (5) and **A4** we obtain

$$
\begin{aligned}
D^{+}V(t) \leq \sum_{i=1}^{m} \Bigg\{ &(\lambda - \underline{\alpha}_i \underline{\beta}_i) y_i(t) + \overline{\alpha}_i \sum_{j=1}^{m} \Big[|a_{ij}| F_j y_j(t) \\
+ \quad & |b_{ij}| G_j y_j(t - \tau_{ij}(t)) e^{\lambda \tau_{ij}(t)} + |c_{ij}| H_j \int_{-\infty}^{0} e^{-\lambda\theta} y_j(t+\theta)\, d\eta_{ij}(\theta) \Big] \\
+ \quad & \overline{\alpha}_i \sum_{j=1}^{m} |b_{ij}| G_j e^{\lambda \tau_{ij}} \left[\frac{y_j(t)}{1-\delta} - y_j(t - \tau_{ij}(t)) \frac{1 - \dot{\tau}_{ij}(t)}{1-\delta} \right] \\
+ \quad & \overline{\alpha}_i \sum_{j=1}^{m} |c_{ij}| H_j \int_{-\infty}^{0} e^{-\lambda\theta} \big[y_j(t) - y_j(t+\theta) \big] \, d\eta_{ij}(\theta) \Bigg\} \xi_i
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \sum_{j=1}^{m} y_j(t) \sum_{i=1}^{m} \left\{ (\lambda - \underline{\alpha}_i \underline{\beta}_i)\delta_{ij} + \overline{\alpha}_i \left[|a_{ij}| F_j + |b_{ij}| G_j \frac{e^{\lambda \tau_{ij}}}{1-\delta} + |c_{ij}| H_j k_{ij}(\lambda) \right] \right\} \xi_i \\
+ \quad & \sum_{i=1}^{m} \overline{\alpha}_i \xi_i \sum_{j=1}^{m} |b_{ij}| G_j y_j(t - \tau_{ij}(t)) \left(e^{\lambda \tau_{ij}(t)} - e^{\lambda \tau_{ij}} \frac{1 - \dot{\tau}_{ij}(t)}{1-\delta} \right) \leq 0.
\end{aligned}
$$

This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$
V(t) \leq V(t_{k-1} + 0) \quad \text{for} \quad t_{k-1} < t \leq t_k. \tag{11}
$$

In particular,

$$
V(t_k) \leq V(t_{k-1} + 0), \qquad k \in \mathbb{N}. \tag{12}
$$

Further on, for $k \in \mathbb{N}$ we find successively

$$
w_i(t_k + 0, x) = (1 - B_{ik}) w_i(t_k, x) + \int_{t_{k-1} - t_k}^{0} w_i(t_k + \theta, x) \, d\zeta_k(\theta),
$$

$$
\|w_i(t_k + 0, \cdot)\| \leq |1 - B_{ik}| \, \|w_i(t_k, \cdot)\| + \int_{t_{k-1} - t_k}^{0} \|w_i(t_k + \theta, \cdot)\| \, d\zeta_k(\theta)
$$

and

$$
y_i(t_k + 0) \leq |1 - B_{ik}| y_i(t_k) + \int_{t_{k-1} - t_k}^{0} e^{-\lambda\theta} y_i(t_k + \theta) \, d\zeta_k(\theta).
$$

Making use of (11) and (12), we obtain

$$
\begin{aligned}
V(t_k + 0) \quad & \leq \quad \max_{i=\overline{1,m}} |1 - B_{ik}| V(t_k) + \int_{t_{k-1} - t_k}^{0} e^{-\lambda\theta} \, d\zeta_k(\theta) \, V(t_{k-1} + 0) \\
& \leq \quad \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{t_{k-1} - t_k}^{0} e^{-\lambda\theta} \, d\zeta_k(\theta) \right) V(t_{k-1} + 0).
\end{aligned}
$$

Combaning the last estimate with (11), (12) and (10), we derive (6) □

For three sets of additional assumptions we will show that inequality (6) implies global exponential stability of the equilibrium point u^* of the impulsive system (1).

Corollary 1 *Let all conditions of Theorem 1 hold and*

$$
\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{t_{k-1} - t_k}^{0} e^{-\lambda\theta} \, d\zeta_k(\theta) \leq 1
$$

for all sufficiently large values of $k \in \mathbb{N}$. *Then the equilibrium point* u^* *of the impulsive system* (1) *is globally exponentially stable with Lyapunov exponent* λ.

In the above corollary the global exponential stability was provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that those do not occur too often.

Corollary 2 *Let all conditions of Theorem 1 hold and*

$$\limsup_{t\to\infty} \frac{i(0,t)}{t} = p < +\infty.$$

Let there exist a positive constant B satisfying the inequalities

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \leq B$$

and $p \ln B < \lambda$. Then for any $\tilde{\lambda} \in (0, \lambda - p\ln B)$ the equilibrium point u^* of the impulsive system (1) *is globally exponentially stable with Lyapunov exponent $\tilde{\lambda}$.*

Similar conditions were introduced in our previous paper [1].

Corollary 3 *Let all conditions of Theorem 1 and there exists a constant $\kappa \in (0,\lambda)$ satisfying the inequality*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \leq e^{\kappa(t_k - t_{k-1})}$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point u^* of the impulsive system (1) *is globally exponentially stable with Lyapunov exponent $\lambda - \kappa$.*

A similar condition was introduced in the paper [15].

Acknowledgement. This work was partially supported by the collaborative research project CL/SQU–UAEU/10/02 between Sultan Qaboos University and the UAE University.

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